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An Example of Bifurcation to Homoclinic Orbits

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Consider the equation $\ddot{x} - x + x^2 = -\lambda_1\dot{x} + \lambda_2 f(t)$ where $f(t+1) = f(t)$ and $\lambda = (\lambda_1, \lambda_2)$ is small. For $\lambda = 0$, there is a homoclinic orbit Γ through zero. For $\lambda \neq 0$ and small, there can be "strange" attractors near Γ. The purpose of this paper is to determine the curves in λ-space of bifurcation to "strange" attractors and to relate this to hyperbolic subharmonic bifurcations.

I. INTRODUCTION

At the present time, there is considerable interest in the theory of "strange" attractors of mappings and the manner in which strange attractors arise through successive bifurcations of periodic orbits (see [1, 5–12]). The simplest "strange" attractors arise near homoclinic points.

For a hyperbolic fixed point p of a diffeomorphism, a point $q \neq p$ is a homoclinic point of p if q is in the intersection of the stable and unstable manifolds of p. The point q is called a transverse homoclinic point of p if the intersection of the stable and unstable manifolds of p at q is transversal. It was known to Poincaré that the existence of one transverse homoclinic point implies there must be infinitely many transverse homoclinic points. Birkhoff showed that every

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transverse homoclinic point is the limit of periodic orbits. Smale proved that the existence of a transverse homoclinic point q of p implies the existence of an invariant Cantor set in the set of nonwandering points and some iterate of the map restricted to this set is the same as the shift automorphism on doubly infinite sequences (see [11, 12] for a bibliography and a proof of these facts). Later it was shown by Newhouse [9] that generically there must exist infinitely many periodic orbits which are asymptotically stable either as $t \to +\infty$ or $-\infty$ near the bifurcation to any transverse homoclinic point of p. Newhouse and Palis [10] have given some information about the manner in which a transverse homoclinic point can arise through successive bifurcations.

From the abstract point of view, the theory of transverse homoclinic points of diffeomorphisms is fairly well understood. However, this theory is not very accessible to persons in the applications. Several papers have been devoted to the discussion of specific examples which illustrate some properties of homoclinic points; for example, [5–8]. The purpose of this paper is to give another example of this type. The papers [5, 7, 8], are similar in spirit to the approach taken here. One of our objectives is to show that the analytic conditions for homoclinic bifurcation automatically imply conditions for subharmonic bifurcation in a neighborhood of the homoclinic orbit. It is primarily this aspect of our work which makes the presentation more complete than the ones mentioned above. After bifurcation to homoclinic points has occurred, a deeper understanding of the behavior requires knowledge of the abstract dynamical systems and symbolic dynamics.

A special case of the results of this paper concerns the example

$$\ddot{x} - x + x^2 = -\lambda_1 \dot{x} + \lambda_2 f(t), \tag{1.1}$$

where $f(t+1) = f(t)$ is continuous, and λ_1, λ_2 are small parameters. For $\lambda = (\lambda_1, \lambda_2) = (0, 0)$, the equilibrium point $(x, \dot{x}) = (0, 0)$ is a saddle point, there is a separatrix Γ with solutions approaching $(0, 0)$ as $t \to \pm\infty$ and a center inside the separatrix. For λ small, there is a unique 1-periodic solution near $(x, \dot{x}) = (0, 0)$. If $T_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ is the time one map of the solutions of (1.1), then $T_0 \Gamma = \Gamma$ and each point of Γ is homoclinic to $(0, 0)$. For λ small, there is a unique fixed point z_λ of T_λ near $(x, \dot{x}) = (0, 0)$ which is hyperbolic. Under certain generic conditions on $f(t)$, we give below a complete description of the regions in the parameter space λ near $\lambda = 0$ in which there is either no point homoclinic to z_λ or points homoclinic to z_λ. These regions are separated by smooth curves C_1, C_2 which are called the curves of bifurcation for homoclinic points.

Inside the separatrix Γ, there are periodic orbits with period approaching ∞ as one approaches Γ. Therefore, one can investigate conditions on the parameters λ to ensure the existence of subharmonics of order k for any integer k; that is, one can seek the curves C_1^k, C_2^k of bifurcation to subharmonics of order k.

Hale end Taboas [2] (see also, Hale [3]) have discussed this problem in detail and gave generic conditions on f for the determination of these curves. It is shown below that the generic conditions on f which ensure the existence of the bifurcation curves C_1, C_2 for homoclinic points implies the conditions in [2] for the existence of the bifurcation curves $C_1{}^k$, $C_2{}^k$ for subharmonics of order k for k large and $C_j{}^k \to C_j$ as $k \to \infty$. Therefore, in any neighborhood of the bifurcation curves C_j for homoclinic orbits, there are infinitely many subharmonic bifurcations. The subharmonic solutions are always hyperbolic—half of them being hyperbolic nodes or foci and half being saddles.

As one approaches the bifurcation curves C_1, C_2 along a smooth path in parameter space, there may or may not occur subharmonic bifurcations. For a fixed value of λ_2 and sufficiently large λ_1, there will be no homoclinic points for (1.1). As λ_1 is decreased, homoclinic points will appear when either the curve C_1 or C_2 is crossed. However, it is possible to have the subharmonic bifurcations occur either before or after the homoclinic points appear. Conditions are given on f to ensure which case prevails. A specific example is given in Section 5 showing that all of the conditions imposed can be satisfied. We are indepted to K. Meyer and D. Henry for these computations.

Let the separatrix be denoted $\Gamma = \{(p(t)\,\dot{p}(t)),\ t \in (-\infty, \infty)\} \cup \{(0, 0))\}$ where p satisfies $\ddot{p} - p + p^2 = 0$. There is no loss in generality to assume $\dot{p}(0) = 0$. The function

$$h(\alpha) = \int_{-\infty}^{\infty} \dot{p}(t) f(t - \alpha)\, dt, \tag{1.2}$$

$h(\alpha + 1) = h(\alpha)$ for all α, was shown by Mel'nikov [7] to play a fundamental role in the theory of homoclinic bifurcation. We impose the following genereic and computable condition:

$$\text{If} \quad h(\alpha_M) = \max h(\alpha), \qquad h(\alpha_m) = \min h(\alpha),$$
$$\text{then} \quad h''(\alpha_M) < 0, \qquad h''(\alpha_m) > 0.$$

Under hypothesis (1.3), the bifurcation curves C_1, C_2 for homoclinic points are tangent to the curves $\lambda_1 = h(\alpha_m)\,\lambda_2$, $\lambda_1 = h(\alpha_M)\,\lambda_2$, respectively, at $\lambda = 0$.

Whether subharmonic bifurcations occur before or after bifurcation to homoclinic points depends upon the function

$$h(\alpha, \delta) = \int_{-\omega(\delta)/2}^{\omega(\delta)/2} \dot{p}_\delta(t) f(t - \alpha)\, dt,$$

where p_δ is an $\omega(\delta)$-periodic solution of $\ddot{x} - x + x^2 = 0$ with $p_\delta(0) = (1 - \delta)\,p(0)$, $\dot{p}_\delta(0) = 0$. In fact, if (1.2) is satisfied, then, for δ sufficiently small, $h(\alpha, \delta)$ has a maximum at $\alpha_M(\delta) \in [0, 1)$ and a minimum at $\alpha_m(\delta) \in [0, 1)$ with $\alpha_M(\delta) \to \alpha_M$, $\alpha_m(\delta) \to \alpha_m$ as $\delta \to 0$ and $h''(\alpha_M(\delta), \delta) < 0$, $h''(\alpha_m(\delta), \delta) > 0$.

Then the subharmonic bifurcations occur before the bifurcation to homoclinic points if $dh(\alpha_M(\delta), \delta)/d\delta > 0$, $dh(\alpha_m(\delta), \delta) < 0$ at $\delta = 0$ and after if the signs are reversed.

The above results are obtained using only ideas encountered in a standard course in differential equations which has covered small parameter methods in nonlinear oscillations. We first obtain an analogue of the Fredholm alternative for bounded solutions of the nonhomogeneous linear variational equation for Γ. The method of Liapunov–Schmidt is then used to obtain the bifurcation equations which are analyzed by elementary applications of the Implicit Function Theorem. Some detailed convergence properties of solutions near a saddle point are then used to obtain the conclusions about subharmonic bifurcations.

The study of the intersection of stable and unstable manifolds has nothing to do with the fact that $f(t)$ is a periodic function. In fact, it is almost as easy to study the general nonautonomous case for $f(t)$ bounded and the intersection in $(x, \dot{x}, t)$-space of the stable and unstable manifolds of the bounded trajectory $\gamma(\lambda, f)$ of (1.1) which for $\lambda = 0$ is $(0, 0) \times \mathbb{R}$. This is certainly more relaistic from the physical point of view. For example, if $f(t)$ is almost periodic, one can show that much of the structure of the periodic case is preserved. This will be discussed in a later paper.

In studying the general nonautonomous case, one can also encounter situations which are not homoclinic in nature in the sense that the stable and unstable manifolds in $(x, \dot{x}, t)$-space intersect in a finite number of curves or even a single curve. When it causes no excessive difficulties, the general case is treated below, but we will concentrate on those problems which have the character of homoclinic points.

2. The Bifurcation Equations

Consider the second order equation

$$\ddot{x} + g(x) = 0 \tag{2.1}$$

or the corresponding system

$$\dot{x} = y, \qquad \dot{y} = g(x), \tag{2.2}$$

where g has continuous derivatives up through order two,

$$g(0) = 0, \qquad g'(0) < 0, \tag{2.3}$$

and $g'(x) = dg(x)/dx$. Hypothesis (2.3) implies the equilibrium point $(0, 0)$ is a saddle point with local stable manifold given approximately by $y = -\gamma x$ and local unstable manifold given approximately by $y = \gamma x$. The characteristic exponents of the equilibrium point are $\pm\gamma$, where $\gamma = [-g'(0)]^{1/2}$.

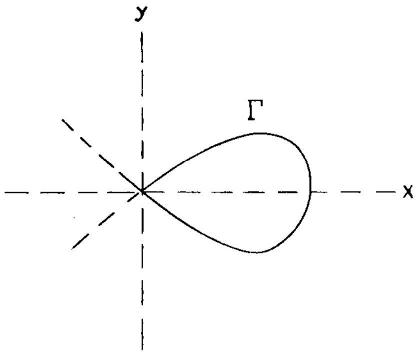


FIGURE 1

Assume the global stable and unstable manifolds of (0, 0) coincide; that is, there is an orbit Γ whose α- and ω-limit sets are (0, 0) as shown in Fig. 1. Such an orbit Γ is called a *homoclinic orbit*. Suppose $\Gamma = \{(p(t), \dot{p}(t)), t \in (-\infty, \infty)\}$ where $\ddot{p} + g(p) = 0$. Let $\bar{\Gamma}$ be the closure of Γ.

Let $\Omega \subset \mathbb{R}^2$ be an open set containing $\bar{\Gamma}$, Λ an open set in $\mathbb{R}^k$ containing zero and $\bar{\Omega}$, $\bar{\Lambda}$ be the closures of Ω, Λ. Suppose $f(t, z, \lambda)$, $(t, z, \lambda) \in \mathbb{R} \times \bar{\Omega} \times \bar{\Lambda}$, has continuous bounded derivatives up through order two in z, λ and let

$$|f(\cdot, \cdot, \lambda)|_2 = \sup\{|\partial^j f(t, z, \lambda)/\partial z^j|, j = 0, 1, 2; (t, z) \in \mathbb{R} \times \bar{\Omega}\}.$$

Our objective is to study the behavior of the solutions of the perturbed equation

$$\ddot{x} + g(x) = f(t, x, \dot{x}, \lambda), \tag{2.4}$$

in a neighborhood of $\bar{\Gamma}$ for λ in a neighborhood of zero for $|f(\cdot, \cdot, \lambda)|_2 \to 0$ as $\lambda \to 0$.

To describe the problem more precisely, we must consider the complete trajectories in (x, y, t)-space of the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) + f(t, x, y, \lambda). \end{aligned} \tag{2.5}$$

The set $\bar{\Gamma} \times \mathbb{R}$ is an invariant set for (2.2) similar to a "cylinder." For any $t_0 \in \mathbb{R}$, $(x_0, y_0) \neq (0, 0)$, $(x_0, y_0, t_0) \in \bar{\Gamma} \times \mathbb{R}$, the trajectory through (x_0, y_0, t_0) twists once around the "cylinder" $\bar{\Gamma} \times \mathbb{R}$ going to zero as $t \to \pm\infty$(see Fig. 2). Each trajectory is given by $(p(t - t_0 + \alpha), \dot{p}(t - t_0 + \alpha), t - t_0)$, where α is determined uniquely by (x_0, y_0) so that $(p(\alpha), \dot{p}(\alpha)) = (x_0, y_0)$.

Since (0, 0) is hyperbolic, the standard theory of differential equations (see, for example, [4]) implies there are $\delta > 0$, $\eta > 0$,such that $|\lambda| < \delta$ implies

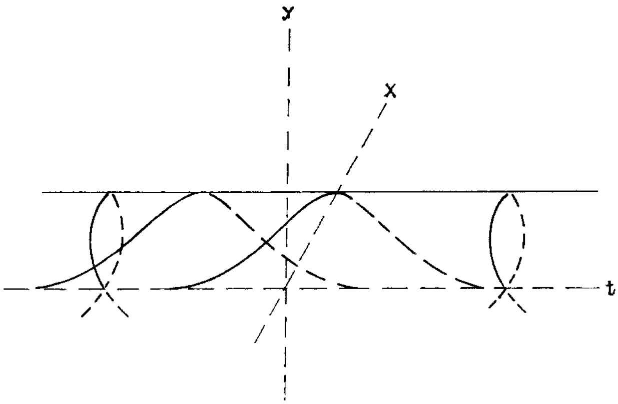


FIGURE 2

there is a unique solution $(\phi(t, \lambda), \partial\phi(t, \lambda)/\partial t)$ of (2.5) which is bounded and satisfies $|\phi(t, \lambda)| < \eta$ for $t \in \mathbb{R}$, $|\lambda| < \delta$, $\phi(t, 0) = 0$. Furthermore, $\phi(t, \lambda)$ is as smooth in λ as the perturbation function $f(t, x, y, \lambda)$.

If $\gamma(\lambda) = \{(\phi(t, \lambda), \partial\phi(t, \lambda)/\partial t, t), t \in \mathbb{R}\} \subset \mathbb{R}^3$, then it is also known that there exist a unique stable manifold $\gamma_s(\lambda) \subset \mathbb{R}^3$ of $\gamma(\lambda)$ and a unique unstable manifold $\gamma_u(\lambda) \subset \mathbb{R}^3$ of $\gamma(\lambda)$. The local stable and unstable manifolds follow from [4]. The global manifolds are obtained in the usual way by extension using the differential equation. These manifolds $\gamma_s(\lambda)$, $\gamma_u(\lambda)$ inherit properties from the perturbation term. For example, $\gamma_s^{t_0}(\lambda) = \{(x, y) : (x, y, t_0) \in \gamma_s(\lambda)\}$ is periodic (almost periodic) in t_0 if $f(t, x, y, \lambda)$ is periodic (almost periodic) in t.

PROBLEM. For a given neighborhood U of Γ and a given neighborhood V of $\lambda = 0$, when is

$$S_\lambda \stackrel{\text{def}}{=} [\gamma_s(\lambda) \cap \gamma_u(\lambda)] \cap (U \times \mathbb{R}) \backslash \gamma(\lambda) \neq \varnothing \qquad \text{for} \quad \lambda \in V?$$

That is, when is there a trajectory $\{(\psi(t, \lambda), \partial\psi(t, \lambda)/\partial t, t) : t \in \mathbb{R}\} \subset U \times \mathbb{R}$ of (2.5) for $\lambda \in V$ such that $\psi(t, \lambda) \neq \phi(t, \lambda)$ and $\psi(t, \lambda) - \phi(t, \lambda) \to 0$ as $t \to \pm\infty$.

If there exists a $\lambda \in V$ such that $S_\lambda \neq \varnothing$ and $f(t, x, y, \lambda)$ is periodic in t, then there will be infinitely many points in $S_\lambda^{t_0} = \{(x, y) : (x, y, t_0) \in S_\lambda\}$ for any t_0 since $S_\lambda^{t_0}$ is periodic in t_0. If a point in S_λ corresponds to a transverse intersection, one obtains infinitely many transverse intersections. Since the stable and unstable manifolds of the time one map are invariant, one obtains the complicated intersection of the stable and unstable manifolds shown in Fig. 3. As remarked in the Introduction, the general theory of dynamical systems gives a more complete explanation of the behavior of the period map—the existence of infinitely many asymptotically stable periodic orbits, for example.

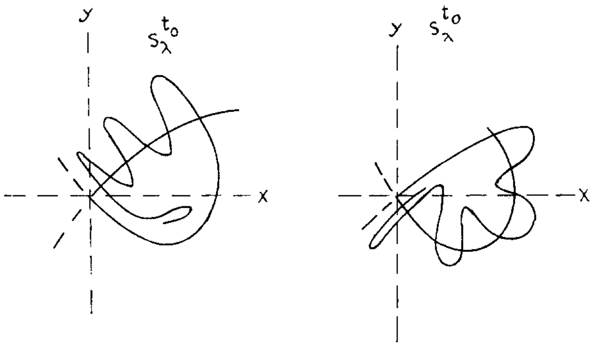


FIGURE 3

In the next few pages, we will obtain necessary and sufficient conditions on λ to ensure that $S_\lambda \neq \varnothing$ and this will be done for general perturbations bounded in t rather than just periodic in t. With the additional assumption of periodicity in t, we will then prove that subharmonic bifurcations are always present near a homoclinic point—half of them being saddles and the other half being asymptotically stable either as $t \to \infty$ or as $t \to -\infty$.

When the perturbation is almost periodic in t, the set $S_\lambda^{t_0} = \{(x, y): (x, y, t_0) \in S_\lambda\}$ is almost periodic in t_0. If we suppose certain generic conditions on all elements of the hull of f, it can be shown that there are infinitely many transverse intersections of $\gamma_u(\lambda)$ and $\gamma_s(\lambda)$.

To obtain necessary and sufficient conditions that $S_\lambda \neq \varnothing$, we first obtain conditions for the existence of solutions bounded on $\mathbb{R}$ for the nonhomogeneous linear variational equation around Γ. More specifically, we prove the following version of the Fredholm alternative for solutions bounded on $\mathbb{R}$. Let $\mathscr{B}(\mathbb{R}) = \{F: \mathbb{R} \to \mathbb{R}$ bounded, continuous$\}$ with $|F| = \sup_{t \in \mathbb{R}} |F(t)|$ for $F \in \mathscr{B}(\mathbb{R})$.

LEMMA 2.1. *If $F \in \mathscr{B}(\mathbb{R})$, the equation*

$$\ddot{z} + g'(p(t))\, z = F(t) \tag{2.6}$$

has a solution in $\mathscr{B}(\mathbb{R})$ if and only if

$$\int_{-\infty}^{\infty} \dot{p}(t) F(t)\, dt = 0, \tag{2.7}$$

that is, if and only if $PF = 0$, where $P: \mathscr{B}(\mathbb{R}) \to \mathscr{B}(\mathbb{R})$ is the continuous projection defined by

$$PF = \frac{1}{\eta} \dot{p} \int_{-\infty}^{\infty} \dot{p} F, \qquad \eta = \int_{-\infty}^{\infty} \dot{p}^2. \tag{2.8}$$

If $PF = 0$, there is a unique solution $\mathscr{K}F$ with initial value orthogonal to $(\dot{p}(0), \ddot{p}(0))$, $\mathscr{K}: (I - P)\mathscr{B}(\mathbb{R}) \to \mathscr{B}(\mathbb{R})$ is continuous and linear.

Proof. Although the proof follows standard arguments, it is included for completeness. The homogeneous equation

$$\ddot{z} + g'(p(t))\, z = 0 \tag{2.9}$$

has the solution $\dot{p}(t)$ since $\ddot{p} + g(p) = 0$. Let q be a solution of (2.9) such that the Wronskian of $(\dot{p}, q)$ is one, that is,

$$\dot{p}\dot{q} - \ddot{p}q = 1, \tag{2.10}$$

and choose $q(0) = -1/\ddot{p}(0)$, $\dot{q}(0) = 0$. The variation of constants formula implies that every solution of (2.6) is given by

$$z(t) = Aq(t) + B\dot{p}(t) + \int_0^t [\dot{p}(s)\, q(t) - \dot{p}(t)\, q(s)]\, F(s)\, ds, \tag{2.11}$$

where A, B are arbitrary constants.

Since $(\dot{p}, \ddot{p})$ is the tangent vector to Γ, it is the tangent vector to the stable and unstable manifolds of the equilibrium point $(0, 0)$ of (2.2). Thus, if $\gamma = [-g'(0)]^{1/2}$, then

$$\begin{aligned} (\dot{p}(t), \ddot{p}(t))\, e^{\gamma t} &\to \text{constant} \qquad \text{as } t \to \infty, \\ (\dot{p}(t), \ddot{p}(t))\, e^{-\gamma t} &\to \text{constant} \qquad \text{as } t \to -\infty, \end{aligned}$$

which implies from (2.10) that

$$\begin{aligned} (q(t), \dot{q}(t))\, e^{-\gamma t} &\to \text{constant} \qquad \text{as } t \to \infty, \\ (q(t), \dot{q}(t))\, e^{\gamma t} &\to \text{constant} \qquad \text{as } t \to -\infty. \end{aligned}$$

Thus, there is a constant K such that

$$\begin{aligned} |\dot{p}(t)\, q(s)| &\leqslant Ke^{-\gamma(t-s)} \qquad \text{for} \quad t, s \geqslant 0, \\ |\dot{p}(t)\, q(s)| &\leqslant Ke^{\gamma(t-s)} \qquad \text{for} \quad t, s \leqslant 0. \end{aligned}$$

From these relations it is now easy to verify that $z(t)$ in (2.11) is bounded on $[0, \infty)$ if and only if

$$A = -\int_0^\infty \dot{p}(s)\, F(s)\, ds \tag{2.12}$$

and bounded on $(-\infty, 0]$ if and only if

$$A = \int_{-\infty}^0 \dot{p}(s)\, F(s)\, ds. \tag{2.13}$$

Thus, $z(t)$ in (2.11) is bounded on $\mathbb{R}$ if and only if (2.7) is satisfied and then

$$z(t) = B\dot{p}(t) - \dot{p}(t) \int_0^t q(s) F(s)\, ds + q(t) \int_\infty^t \dot{p}(s) F(s)\, ds. \tag{2.14}$$

To have $(z(0), \dot{z}(0))$ orthogonal to $(\dot{p}(0), \ddot{p}(0))$, we must have

$$B[\dot{p}(0)^2 + \ddot{p}(0)^2] = -\dot{p}(0)\, \dot{q}(0) \int_\infty^0 \dot{p}(s) F(s)\, ds. \tag{2.15}$$

This uniquely defines B as a continuous linear functional on $\mathscr{B}(\mathbb{R})$. From (2.14), one observes that the corresponding solution $z =^{\mathrm{def}} \mathscr{K}F$ is continuous and linear on $\mathscr{B}(\mathbb{R})$. This proves the lemma.

To obtain necessary and sufficient conditions that $S_\lambda \neq \varnothing$, we must understand some more about the stable and unstable manifolds for $\gamma(\lambda)$.

If x is a solution of (2.4),

$$x(\tau) = p(\tau + \alpha) + z(\tau + \alpha), \qquad \tau + \alpha = t, \tag{2.16}$$

then

$$\begin{aligned} &\ddot{z} + g'(p(t))\, z = F(t, z, \dot{z}, \lambda, \alpha) \\ &F(t, z, \dot{z}, \lambda, \alpha) \overset{\mathrm{def}}{=} f(t - \alpha, p(t) + z, \dot{p}(t) + z, \lambda) \\ &\qquad\qquad - g(p(t) + z) + g(p(t)) + g'(p(t))\, z. \end{aligned} \tag{2.17}$$

Let us now consider all solutions of (2.17) which are bounded as $t \to -\infty$. The solution z is given by (2.11) with A satisfying (2.13); that is,

$$\begin{aligned} z(t) = B\dot{p}(t) &- \dot{p}(t) \int_0^t q(s) F(s, z(s), \dot{z}(s), \lambda, \alpha)\, ds \\ &+ q(t) \int_{-\infty}^t \dot{p}(s) F(s, z(s), \dot{z}(s), \lambda, \alpha)\, ds. \end{aligned} \tag{2.18}$$

In particular, the solution $\phi(t, \lambda)$ which describes the curve $\gamma(\lambda)$ is a solution as well as the solutions which approach γ as $t \to -\infty$; that is, all those solutions with initial values on $\gamma_u(\lambda)$. One can actually prove (see, for example, [4])] that the only solutions of (2.18) which remain in a sufficiently small neighborhood of zero as $t \to -\infty$ must have initial values on $\gamma_u(\lambda)$. In the same way, one obtains the stable manifold $\gamma_s(\lambda)$ from the solutions which remain in a sufficiently small neighborhood of zero as $t \to \infty$ of the equation

$$\begin{aligned} z(t) = B\dot{p}(t) &- \dot{p}(t) \int_0^t q(s) F(s, z(s), \dot{z}(s), \lambda, \alpha)\, ds \\ &+ q(t) \int_\infty^t \dot{p}(s) F(s, z(s), \dot{z}(s), \lambda, \alpha)\, ds. \end{aligned} \tag{2.19}$$

Thus, $\gamma_u(\lambda) \cap \gamma_s(\lambda) \backslash \gamma(\lambda)$ has elements in a neibhborhood of Γ if and only if both (2.18) and (2.19) are satisfied; that is, from Lemma 2.1,

$$PF(\cdot, z, \dot{z}, \lambda, \alpha) = 0, \tag{2.20a}$$

$$z = \mathscr{K}(I - P)F(\cdot, z, \dot{z}, \lambda, \alpha). \tag{2.20b}$$

An application of the Implicit Function Theorem shows there is a $\delta > 0$ and a unique solution $z^*(\lambda, \alpha)$ of (2.20b) for $|\lambda| < \delta$, $|z| < \delta$, $\alpha \in \mathbb{R}$, $z^*(\lambda, \alpha)$ has continuous derivatives up through order two in λ, α, $z^*(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$. Using the definition of P, we have proved the following theorem.

THEOREM 2.1. *Suppose g, f satisfy the conditions above and $z^*(\lambda, \alpha)$ is the solution of* (2.20b). *Then there is a $\delta > 0$, $\lambda_0 > 0$, such that* (2.4) *has a unique solution $\phi(t, \lambda)$, $t \in \mathbb{R}$, $|\lambda| < \lambda_0$, with $|\phi(t, \lambda)| < \delta$, $\phi(t, 0) = 0$. Furthermore, there is a solution $x(t, \lambda)$, $t \in \mathbb{R}$, $|\lambda| < \lambda_0$, of* (2.4) *distinct from $\phi(t, \lambda)$ such that*

$$\begin{aligned} &d(x(t, \lambda), \Gamma) < \delta, \\ &|x(t, \lambda) - \phi(t, \lambda)| \to 0 \quad \text{as} \quad t \to \pm\infty \end{aligned} \tag{2.21}$$

if and only if $x(\cdot, \lambda) = p(\cdot + \alpha) + z^*(\lambda, \alpha)$, $x(t, 0) = p(t + \alpha)$ and (λ, α) satisfy the equation

$$\begin{aligned} &G(\lambda, \alpha) = 0, \\ &G(\lambda, \alpha) \overset{\text{def}}{=} \frac{1}{\eta} \int_{-\infty}^{\infty} \dot{p}(t) F(t, z^*(\lambda, \alpha)(t), \dot{z}^*(\lambda, \alpha)(t), \lambda, \alpha)\, dt, \\ &\eta = \int_{-\infty}^{\infty} \dot{p}^2. \end{aligned} \tag{2.22}$$

Finally, the function $G(\lambda, \alpha)$ has continuous derivatives up through order two in λ, α, $G(0, \alpha) = 0$.

It remains to analyze the solutions of (2.22) for λ small and $a \in \mathbb{R}$. The remaining sections are devoted to special cases for the function f in (2.4).

Before doing this, we make a few further remarks.

Remark 2.1. Theorem 2.1 asserts that the points $q = p(\alpha_0) + z^*(\lambda_0, \alpha_0)(0)$ are homoclinic to the point $\phi(0, \lambda_0)$ if and only if $G(\lambda_0, \alpha_0) = 0$. It is also true that q is transverse homoclinic to $\phi(0, \lambda_0)$ if and only if $G(\lambda_0, \alpha_0) = 0$ and $\partial G(\lambda_0, \alpha_0)/\partial \alpha \neq 0$. In fact, if $\partial G(\lambda_0, \alpha_0)/\partial \alpha \neq 0$ and q is not transverse homoclinic to $\phi(0, \lambda_0)$, then one could perturb the vector field in (2.4) by a small function h to obtain a new function $\tilde{\phi}(t, \lambda, h)$ and a new function $\tilde{G}(\lambda, h, \alpha)$ whose zeros yield points homoclinic to $\phi(0, \lambda, h)$. But $\tilde{G}(\lambda, h, \alpha)$ will have a unique solution near the original λ_0, α_0; that is, there is always a point homoclinic to $\tilde{\phi}(0, \lambda, h)$

for λ near λ_0 and h near zero. On the other hand, if the intersection of the stable and unstable manifolds were not transverse, we could choose h so that no homoclinic point exists, which is a contradiction. If the stable and unstable manifolds intersect at a point $q = p(\alpha) + z^*(\lambda, \alpha)\ (0)$ which is transverse to $\phi(0, \lambda_0)$, $G(\lambda_0, \alpha_0) = 0$ and $\partial G(\lambda_0, \alpha_0)/\partial\alpha = 0$, then one could perturb the vector field in (2.4) to eliminate the intersection which would be a contradiction.

Remark 2.2. The above discussion is applicable to the case of a pair of homoclinical orbits. For example, consider the equation

$$\ddot{x} - x + x^3 = 0.$$

We have a pair of homoclinical orbits $p(t)$ and $q(t)$ as shown in Fig. 4.

As before, we obtain bifurcation equations for the perturbed equation

$$\ddot{x} - x + x^3 = \lambda_1 \dot{x} + \lambda_2 f(t), \qquad f(t) = f(t+1).$$

The first order terms in these equations are governed by the functions

$$h(\alpha) = \int_{-\infty}^{\infty} \dot{p}(t) f(t-\alpha)\, dt,$$

$$g(\alpha) = \int_{-\infty}^{\infty} \dot{q}(t) f(t-\alpha)\, dt.$$

These functions will give regions in the $\lambda_1\lambda_2$-space in which all the stable and unstable manifolds of the time one map intersect transversally. (See Fig. 5.)

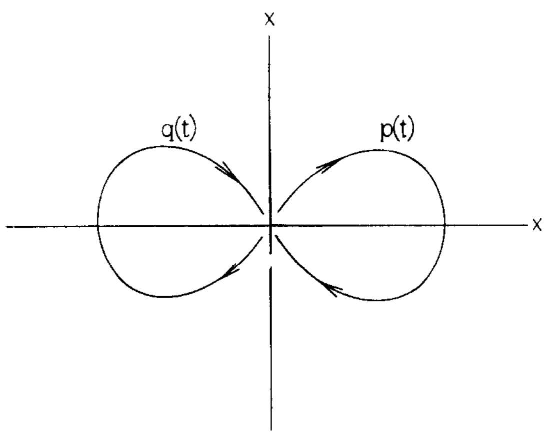


FIGURE 4

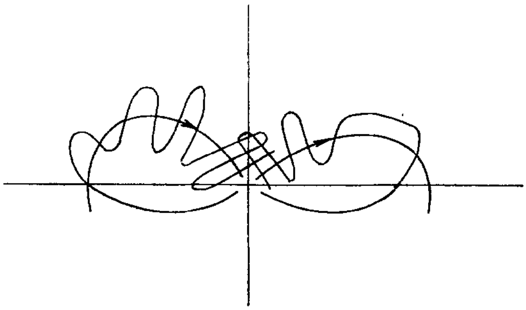


FIGURE 5

Remark 2.3. We also note that if $\lambda_1 = 0$, then we are in fact consiedring area-preserving diffeomorphisms. Our results in this case coincide with those of McGehee and Meyer [6]. However, the proofs are quite different. The advantage to our approach is that we are also able to handle the case where there is slight damping. Moreover, we are able to describe the manner of appearance of homoclinical points through a succession of subharmonic bifurcations. This is done in the next section.

3. Homoclinic Structure

In this section, we consider the special equation

$$\ddot{x} + g(x) = -\lambda_1 \dot{x} + \lambda_2 f(t), \tag{3.1}$$

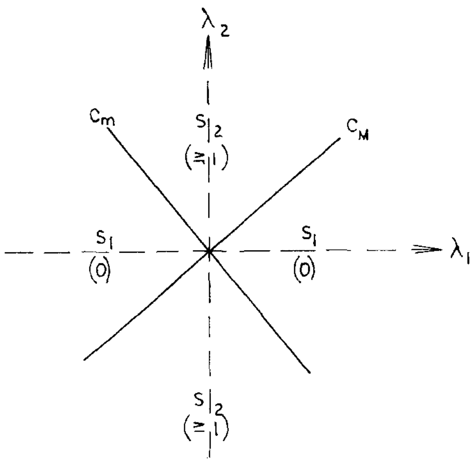
where g satisfies the hypotheses in Section 2, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ is a small parameter, f is continuous and bounded on $\mathbb{R}$. If the homoclinic orbit $\Gamma = \{(p(t), \dot{p}(t)), t \in \mathbb{R}\} \cup \{(0, 0)\}$ and

$$h(\alpha) = \frac{1}{\eta} \int_{-\infty}^{\infty} \dot{p}(t) f(t-\alpha)\, dt, \qquad \eta = \int_{-\infty}^{\infty} \dot{p}^2, \tag{3.2}$$

then we also suppose $h(\alpha)$ has a maximum at α_M, minimum at α_m which satisfy

$$h''(\alpha_M) < 0, \qquad h''(\alpha_m) > 0, \qquad h(\alpha_m) < h(\alpha_M). \tag{3.3}$$

Let $\gamma(\lambda) = \{(\phi(t, \lambda), \dot{\phi}(t, \lambda), t), t \in \mathbb{R}\}$ be the unique bounded trajectory of (3.1) in a neighborhood of zero which satisfies $\gamma(0) = \{(0, 0)\} \times \mathbb{R}$ and let $\gamma_s(\lambda), \gamma_u(\lambda) \in \mathbb{R}^3$ be the stable and unstable manifolds of $\gamma(\lambda)$. We need the following result.



FURIGE 6

LEMMA 3.1. *There are neighborhoods U of Γ, V of $\lambda = 0$ and two C^2-curves C_M, $C_m \subseteq V$ tangent to $\lambda_1 = h(\alpha_M)\,\lambda_2$, $\lambda_1 = h(\alpha_m)\,\lambda_2$, respectively, at $\lambda = 0$ such that C_M, C_m divide the neighborhood V into disjoint sets S_1, S_2 as shown inf Fig. 6 so that $D_\lambda \stackrel{\text{def}}{=} \gamma_s(\lambda) \cap \gamma_u(\lambda) \backslash \gamma(\lambda)$ has no elements in $U \times \mathbb{R}$ if $\lambda \in S_1$ and at least one element if $\lambda \in S_2$.*

Proof. From Theorem 2.1, we need only find the zeros of $G(\lambda, \alpha) = 0$ where G is defined in (2.22). From the special form of f in (3.1), we have

$$G(\lambda, \alpha) = -\lambda_1 + h(\alpha)\,\lambda_2 + G_1(\lambda, \alpha),$$
$$G_1(\lambda, \alpha) = O(|\lambda|^2) \qquad \text{as } |\lambda| \to 0.$$

Finding all solutions of $G(\lambda, \alpha) = 0$ is equivalent to finding all possible solutions of

$$H(\beta, \lambda_2, \alpha) = -\beta + h(\alpha) + G_2(\beta, \lambda_2, \alpha)$$

for $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, λ_2 in a small neighborhood of zero, where $G_2(\beta, \lambda_2, \alpha) = G_1(\beta\lambda_2, \lambda_2, \alpha)/\lambda_2$ satisfies $G_2(\beta, 0, \alpha) = 0$. For $\lambda_2 = 0$, the only possible solutions (β_0, α_0) are those for which $\beta_0 = h(\alpha_0)$. From continuity of the function $H(\beta, \lambda_2, \alpha)$, for any $\epsilon > 0$, there is a $\delta > 0$, such that for any $\beta_0 \in [h(\alpha_m) + \epsilon, h(\alpha_M) - \epsilon]$, there is a solution for $|\lambda| < \delta$. Thus, it remains only to discuss a neighborhood of the points $h_m = h(\alpha_m)$, $h_M = h(\alpha_M)$. Let α_0 be either α_m or α_M.

We have $h'(\alpha_0) = 0$, $h''(\alpha_0) \neq 0$ by our hupothesis on h. Thus, there is a $\delta(\beta_0, \alpha_0)$ and a function $\alpha^*(\beta, \lambda_2)$, $\alpha^*(\beta_0, 0) = \alpha_0$, such that $\partial H(\beta, \lambda_2, \alpha^*(\beta, \lambda_2))/\partial\alpha = 0$ for $|\beta - \beta_0|$, $|\lambda_2| < \delta(\beta_0, \alpha_0)$ and $\alpha^*(\beta, \lambda_2)$ is unique in the region $|\alpha| < \delta(\beta_0, \alpha_0)$. Thus, the function $M(\beta, \lambda_2) = H(\beta, \lambda_2, \alpha^*(\beta, \lambda_2))$

is a maximum or a minimum of $H(\beta, \lambda_2, \alpha)$ with respect to α for β, λ_2 fixed. One easily shows that $M(\beta_0, 0) = 0$, $\partial M(\beta_0, 0)/\partial\beta = -1$. Thus, the Implicit Function Theorem implies there is a $\delta(\beta_0)$ and a unique $\beta^*(\lambda_2)$, $\beta^*(0) = \beta_0$, such that $M(\beta^*(\lambda_2), \lambda_2) = 0$ for $|\lambda| < \delta(\beta_0)$. There are two simple solutions of $H(\beta, \lambda_2, \alpha) = 0$ on one side of the curve $\beta = \beta^*(\lambda_2)$ and no solutions on the other. In terms of the original coordinates (λ_1, λ_2), this implies there are two solutions $\alpha(\lambda)$ one side of the curve $\lambda_1 = \beta^*(\lambda_2)\,\beta_2$ and no solutions on the other, $\alpha(0) = \alpha_0$. The curve $\lambda_1 = \beta^*(\lambda_2)\,\lambda_2$ is a bifurcation curve and is tangent to the line $\lambda_1 = \beta_0\lambda_2$ at $\lambda = 0$.

The above argument also shows that for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$C_m \subseteq \{\lambda_1 = \beta\lambda_2, |\beta - h_m| \leqslant \epsilon, |\lambda_2| < \delta\} = V_m,$$
$$C_M \subseteq \{\lambda_1 = \beta\lambda_2, |\beta - \beta_M| \leqslant \epsilon, |\lambda_2| < \delta\} = V_M,$$

$V_m - C_m = V_m' \cup V_m{}^2$, $V_M - C_M = V_M' \cup V_M{}^2$, such that there are no solutions α of $G(\lambda, \alpha) = 0$ for λ in $V_m' \cup V_M'$ and at least two solutions for λ in $V_m{}^2 \cup V_M{}^2$. This completes the proof of the lemma.

COROLLARY 3.1. *If f in (3.1) satisfies $f(t+1) = f(t)$, hypothesis (3.3), there exists a finite number of points $\alpha_j \in [0, 1)$ such that $h'(\alpha_j) = 0$, and the sector S_2 is defined in Theorem 3.1, then the stable manifold $\gamma_s(\lambda)$ and unstable manifold $\gamma_u(\lambda)$ for $\lambda \in S_2$ have infinitely many transverse intersections in $U \times \mathbb{R}$.*

Proof. This really is a consequence of the proof of Theorem 3.1 and an application of the Implicit Function Theorem to $H(\beta, \lambda_2, \alpha)$ near a point $(\beta_0, 0, \alpha_0)$, $\beta_0 = h(\alpha_0)$, $h'(\alpha_0) \neq 0$ since $h(\alpha + 1) = h(\alpha)$ for all α.

If the conditions of Corollary 3.1 are satisfied, it follows that the time one map for (3.1) has infinitely many transverse homoclinic points and the complicated structure described in Section 2.

4. SUBHARMONIC BIFURCATION

Our next objective is to obtain more details about the manner in which homoclinic orbits appear for $f(t+1) = f(t)$ in (3.1). A subharmonic solution of (3.1) is a k-periodic solution with $k > 1$ an integer. We are especially interested in the way subharmonic bifurcations occur. To do this, we need some more detailed information about some integrals related to $h(\alpha)$ in (3.2).

Recall that we normalized p so that $\dot{p}(0) = 0$. For the equation,

$$\ddot{x} + g(x) = 0, \tag{4.1}$$

let $p_\delta(t)$ be the solution with $p_\delta(0) = (1 - \delta)\,p(0)$, $\dot{p}_\delta(0) = 0$, $0 \leqslant \delta \leqslant \delta_0$,

$\delta_0 > 0$ sufficiently small. Then $p_\delta(t)$, $0 < \delta \leqslant \delta_0$, is periodic of least period $\omega(\delta)$, where $\omega(\delta) \to \infty$ as $\delta \to 0$. Let

$$\begin{aligned} h(\alpha, \delta) &= \int_{-\omega(\delta)/2}^{\omega(\delta)/2} \dot{p}_\delta(t) f(t - \alpha)\, dt, \\ h(\alpha, 0) &= \int_{-\infty}^{\infty} \dot{p}(t) f(t - \alpha)\, dt. \end{aligned} \tag{4.2}$$

If $f(t + 1) = f(t)$, then $h(\alpha + 1, \delta) = h(\alpha, \delta)$, $0 \leqslant \delta \leqslant \delta_0$.

LEMMA 4.1. *For any $f\colon \mathbb{R} \to \mathbb{R}$, continuous, $f(t + 1) = f(t)$ for all t and, for any $\epsilon > 0$, there is a $\delta_1 > 0$ such that*

$$\left| \frac{\partial^j h(\alpha, \delta)}{\partial \alpha^j} - \frac{\partial h(\alpha, 0)}{\partial \alpha^j} \right| \leqslant \epsilon \sup_{0 \leqslant t \leqslant 1} |f(t)|$$

for $0 \leqslant \delta \leqslant \delta_1$, $j = 0, 1, 2$.

Proof. Let $T = \omega(\delta)/2$ and let $\epsilon > 0$ be given. We will show first that there exists a $\delta_1 > 0$ such that $| h(\alpha, 0) - h(\alpha, \delta)| < \epsilon$ for all $0 \leqslant \delta < \delta_1$. Since the level curves are symmetric about the x-axis and $\dot{p}(0) = \dot{p}_\delta(0) = 0$, we only have to show that for all $0 \leqslant \delta < \delta_1$,

$$\left| \int_{-\infty}^{0} \dot{p}(t) f(t - \alpha)\, dt - \int_{-T}^{0} \dot{p}_\delta(t) f(t - \alpha)\, dt \right| < \epsilon/2.$$

By choosing δ_1 sufficiently small, we have

$$\left| \int_{-\infty}^{-T} \dot{p}(t) f(t - \alpha)\, dt \right| < \epsilon/6, \qquad \text{for all } 0 \leqslant \delta < \delta_1 .$$

Fix $0 < \eta < 1$ and

$$2\eta M < \epsilon/6, \qquad \text{where} \quad M = \sup_t |f(t)|.$$

Next, we solve t uniquely, but separately, as a function of x along $p(t)$ and $p_\delta(t)$ for $x > 0$ by using the formula

$$\frac{dx}{dt} = \pm(E - 2G(x))^{1/2},$$

where E is a constant and

$$G(x) = \int_0^x g(t)\, dt.$$

If we let $t(x)$ and $t_\delta(x)$ be the inverse functions, then

$$\begin{aligned}
&\left| \int_{-T}^{0} \dot{p}(t) f(t-\alpha)\, dt - \int_{-T}^{0} \dot{p}_\delta(t) f(t-\alpha)\, dt \right| \\
&\quad = \left| \int_{p(-T)}^{p(0)} f(t(x)-\alpha)\, dx - \int_{p_\delta(-T)}^{p_\delta(0)} f(t_\delta(x)-\alpha)\, dx \right| \\
&\quad \leqslant \left| \int_{p(-T)}^{\eta} f(t(x)-\alpha)\, dx - \int_{p_\delta(-T)}^{\eta} f(t_\delta(x)-\alpha)\, dx \right| \\
&\qquad + \left| \int_{p_\delta(0)}^{p(0)} f(t_\delta(x)-\alpha)\, dx \right| \\
&\qquad + \left| \int_{\eta}^{p(0)} f(t(x)-\alpha)\, dx - \int_{\eta}^{p(0)} f(t_\delta(x)-\alpha)\, dx \right| \\
&\quad \leqslant 2\eta M + |p(0) - p_\delta(0)|\, M \\
&\qquad + \left| \int_{\eta}^{p(0)} f(t(x)-\alpha)\, dx - \int_{\eta}^{p(0)} f(t_\delta(x)-\alpha)\, dx \right|.
\end{aligned}$$

Since η is fixed, $t_\delta(x) \to t(x)$ uniformly for $\eta \leqslant x \leqslant p(0)$ as $\delta \to 0$. We may now decrease δ, if necessary, so that the right-hand side of the above inequality is less than $\epsilon/6$.

Since $\dot{p}(T) = \dot{p}(-T) = 0$, p_δ is C^3 and

$$h(\alpha, \delta) = \int_{-T-\alpha}^{T-\alpha} \dot{p}_\delta(t+\alpha) f(t)\, dt$$

we have

$$\begin{aligned}
\frac{\partial h}{\partial \alpha}(\alpha, \delta) &= \int_{-T-\alpha}^{T-\alpha} \ddot{p}_\delta(t+\alpha) f(t)\, dt, \\
\frac{\partial^2 h}{\partial \alpha^2}(\alpha, \delta) &= -\ddot{p}_\delta(T) f(T-\alpha) + \ddot{p}_\delta(-T) f(-T-\alpha) \\
&\quad + \int_{-T-\alpha}^{T-\alpha} \dddot{p}_\delta(t+\alpha) f(t)\, dt
\end{aligned}$$

and h is C^2 in α. Similarly, we can show the existence of continuous derivatives as specified for $h(\alpha, 0)$. We will show that for every $\epsilon > 0$ there exists a $\delta_2 > 0$ such that

$$\left| \frac{\partial}{\partial \alpha} h(\alpha, 0) - \frac{\partial}{\partial \alpha} h(\alpha, \delta) \right| < \epsilon, \qquad \text{for all } |\delta| < \delta_2 .$$

First, we note that $f(t)$ is periodic. Let $\tilde{f}(t)$ be a smooth periodic approximation of f and

$$\tilde{h}(\alpha, \delta) = \int_{-T-\alpha}^{T-\alpha} \dot{p}_\delta(t+\alpha)\tilde{f}(t)\,dt$$

$$= \int_{-T}^{T} \dot{p}_\delta(t)\tilde{f}(t-\alpha)\,dt$$

Hence, if $\tilde{f}$ is sufficiently close to f,

$$\left| \frac{\partial}{\partial\alpha} h(\alpha, \delta) - \frac{\partial}{\partial\alpha} \tilde{h}(\alpha, \delta) \right| \leqslant \sup_{0\leqslant t\leqslant 1} |f(t) - \tilde{f}(t)| \int_{-T}^{T} |\ddot{p}_\delta(t)|\,dt < \frac{\epsilon}{3}.$$

Similarly, define

$$\tilde{h}(\alpha, 0) = \int_{-\infty}^{\infty} \dot{p}(t+\alpha)\tilde{f}(t)\,dt.$$

Again

$$\left| \frac{\partial}{\partial\alpha} h(\alpha, 0) - \frac{\partial}{\partial\alpha} \tilde{h}(\alpha, 0) \right| \leqslant \sup_{0\leqslant t\leqslant 1} |f(t) - \tilde{f}(t)| \int_{-\infty}^{\infty} |\ddot{p}(t)|\,dt < \frac{\epsilon}{3}.$$

On the other hand,

$$\left| \frac{\partial}{\partial\alpha} \tilde{h}(\alpha, 0) - \frac{\partial}{\partial\alpha} \tilde{h}(\alpha, \delta) \right|$$

$$= \left| \int_{-\infty}^{\infty} \dot{p}(t) \left(\frac{d}{d\alpha}\tilde{f}(t-\alpha)\right) dt - \int_{-T}^{T} \dot{p}_\delta(t) \left(\frac{d}{d\alpha}\tilde{f}(t-\alpha)\right) dt \right|.$$

The first part of our proof show that there exists $\delta_2 > 0$ such that the above expression is less than $\epsilon/3$ for all $0 \leqslant \delta < \delta_2$.

To obtain a similar estimate for the second derivatives of h, we proceed as above taking into account that $\ddot{p}_\delta(-T), \ddot{p}_\delta(T)$ approach zero as $\delta \to 0$. This completes the proof of the lemma.

Lemma 4.1 and hypothesis (3.3) imply there is a $\delta_2 > 0$ such that each function $h(\alpha, \delta)$, $0 \leqslant \delta \leqslant \delta_2$, has a maximum at $\alpha_M{}^\delta$, minimum at $\alpha_m{}^\delta$ such that $\alpha_M{}^\delta \to \alpha_M$, $\alpha_m{}^\delta \to \alpha_m$ as $\delta \to 0$ and there is an $\eta_0 > 0$ such that

$$\partial^2 h(\alpha_M{}^\delta, \delta)/\partial\alpha^2 < -\eta_0 < 0, \qquad \partial^2 h(\alpha_m{}^\delta, \delta)/\partial\alpha^2 > \eta_0 > 0, \tag{4.3}$$

for $0 \leqslant \delta \leqslant \delta_2$.

In the following, we will also assume that

$$d\omega(\delta)/d\delta > 0, \qquad 0 < \delta \leqslant \delta_0. \tag{4.4}$$

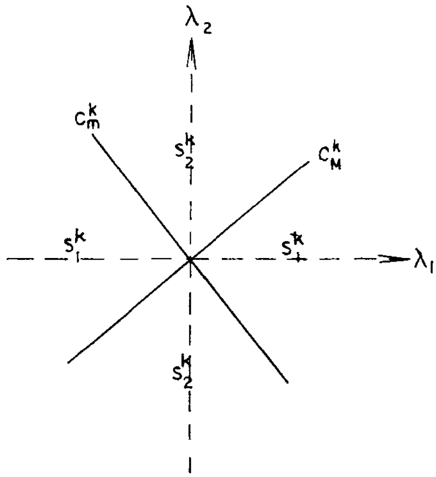


FIGURE 7

Since $\omega(\delta) \to \infty$ as $\delta \to 0$, there is a k_0 such that for any integer $k \geqslant k_0$, there is a $\delta_k \in (0, \delta_2]$, $\delta_k \to 0$ as $k \to \infty$ such that $\omega(\delta_k) = k$; that is, $p_{\delta_k}(t + k) = p_{\delta_k}(t)$ for all $t \in \mathbb{R}$. Furthermore, relation (4.3) will be satisfied for each δ_k.

We can now prove the following result.

LEMMA 4.2. *Suppose f satisfies* (3.3), $f(t + 1) = f(t)$ *and* (4.4) *is satisfied. Then there is a neighborhood U of Γ, V of $\lambda = 0$ and integer $k_0 > 0$, such that, for any $k \geqslant k_0$, there are two C^2-curves $C_M{}^k, C_m{}^k \subseteq V$ tangent to $\lambda_1 = h(\alpha_M^{\delta_k}, \delta_k)\, \lambda_2$, $\lambda_1 = h(\alpha_m^{\delta_k}, \delta_k)\, \lambda_2$, respectively, at $\lambda = 0$ such that $C_M{}^k$, $C_m{}^k$ divide the neighborhood V into disjoint sectors $S_1{}^k$, $S_2{}^k$ as shown in Fig. 7. so that Eq.* (3.1) *has no subharmonics of least period k in $S_1{}^k$ and at least two in $S_2{}^k$. Furthermore, $C_M{}^k$, $C_m{}^k$ approach C_M, C_m as $k \to \infty$, where C_M, C_m are the curves of bifurcation to homoclinic points in Corollary* 3.1.

If, in addition, there are a finite number of points α_j in $[0, 1]$ *such that $h'(\alpha_j) = 0$, where $h(\alpha)$ is given in* (3.2), *and at each of these points $h''(\alpha_j) \neq 0$, then there are at least two subharmonics of least period k in U for each $\lambda \in S_2{}^k$ which are hyperbolic, one is a saddle and the other a node or focus.*

Proof. Choose $k \geqslant k_0$ so that the orbit $\Gamma_k = \{(p_{\delta_k}(t), \dot{p}_{\delta_k}(t)), t \in \mathbb{R}\}$ belongs to the set U in Lemma 3.1. One now applies the results of Hale and Taboas [2] for subharmonic solutions near Γ_k for λ in a neighborhood of zero. If x is a solution of (2.4)

$$x(\tau) = p_{\delta_k}(t + \alpha) + z(\tau + \alpha), \qquad \tau + \alpha = t, \tag{4.5}$$

then

$$\ddot{z} + g'(p_{\delta_k}(t))\, z = F_k(t, z, \dot{z}, \lambda, \alpha) \tag{4.6}$$

$$F(t, z, \dot{z}, \lambda, \alpha) \stackrel{\text{def}}{=} -\lambda_1 \dot{p}_k(t) - \lambda_1 \dot{z} + \lambda_2 f(t - \alpha)$$
$$- g(p_k(t) + z) + g(p_k(t)) + g'(p_k(t))\, z.$$

Hypothesis (4.4) implies the nonhomogeneous linear equation

$$\ddot{z} + g'(p_{\delta_k}(t))\, z = F(t), \tag{4.7}$$

where $F(t + k) = F(t)$ has a k-periodic solution if and only if

$$\int_{-k/2}^{k/2} \dot{p}_{\delta_k}(t)\, F(t)\, dt = 0. \tag{4.8}$$

If

$$P_k F = \dot{p}_{\delta_k} \int_{-k/2}^{k/2} \dot{p}_{\delta_k} F / \eta_k\,, \qquad \eta_k = \int_{-k/2}^{k/2} \dot{p}_{\delta_k}^2\,,$$

and $P_k F = 0$, then there is a unique solution of (4.7), say $\mathscr{K}_k F$, with initial data orthogonal to $(\dot{p}_{\delta_k}(0), \ddot{p}_{\delta_k}(0))$ and $\mathscr{K}_k : (I - P)\mathscr{P}_k \to \mathscr{P}_k$ is continuous and linear, where $\mathscr{P}_k = \{F\colon \mathbb{R} \to \mathbb{R},$ continuous, $F(t + k) = F(t)\}$ with the supremum norm.

As in the proof of Theorem 2.1, the Implicit Function Theorem implies there is a unique solution $z^*(\lambda, \alpha)$ of

$$z = \mathscr{K}_k(I - P_k) F_k(\cdot, z, \dot{z}, \lambda, \alpha)$$

in a neighborhood of $z = 0$, $\lambda = 0$, $\alpha \in \mathbb{R}$. Thus, subharmonic solutions of least period k near Γ_k are given by solutions λ, α of the equation

$$G_k(\lambda, \alpha) = 0,$$
$$G_k(\lambda, \alpha) = \frac{1}{\eta_k} \int_{-k/2}^{k/2} F_k(t, z^*(\lambda, \alpha)(t), \dot{z}^*(\lambda, \alpha)(t), \lambda, \alpha)\, dt$$
$$= -\lambda_1 + \lambda_2 h(\alpha, \delta_k) + \tilde{G}_k(\lambda, \alpha),$$
$$\tilde{G}_k(\lambda, \alpha) = O(|\lambda|^2) \qquad \text{as} \quad |\lambda| \to 0.$$

By Lemma 4.1, all neighborhoods in λ can be chosen independent of $k \geqslant k_0$. Furthermore, we can always restrict the neighborhood of $z = 0$ so that all subharmonic solutions that are obtained in the above way belong to the neighborhood U of Γ in Lemma 3.1.

Lemma 4.1 also implies (4.3) for $k \geqslant k_0$. Now one obtains the bifurcation curves $C_M{}^k$, $C_m{}^k$ as in [2], which is the same proof as that given in Lemma 3.1. It is easy to see that these curves satisfy the properties stated in Lemma 4.2.

It remains to discuss the stability properties of these subharmonic solutions. In a neighborhood of a point $\lambda_0 \neq 0$ on a bifurcation curve $C_M{}^k$ and a neighborhood of α_M^{k0}, $\lambda_{10} = h(\alpha_M^{k0}) \lambda_{20}$, there are either no subharmonic solutions with $x(0)$ near $p_{\delta_k}(\alpha_M^{k0})$ or exactly two solutions. If two solutions exist, they must be hyperbolic. Otherwise, a small perturbation would change a nonhyperbolic one to at least two and we would have more than two solutions, which contradicts the construction of $C_M{}^k$. For any λ, the characteristic multipliers of the linear variational equation of any k-periodic solution satisfy $\mu_1\mu_2 = e^{-k\lambda_1}$. Also, one characteristic multiplier is one at any λ_0 which is a bifurcation point. If $\lambda_0 \neq 0$ is a bifurcation point, then the other characteristic multiplier is <1 if $\lambda_{10} > 0$ and >1 if $\lambda_{10} < 0$.

One now applies the usual procedure of Liapunov–Schmidt for obtaining k-periodic solutions of (3.1) near $p(t + \alpha_M^{k0})$ for λ near λ_0. This is a case where the linear variational equation has one zero characteristic exponent and one with nonzero real part. The bifurcation function is the same as the function $G_k(\lambda, \alpha)$ except restricted to a neighborhood of $(\lambda_0, \alpha_M^{k0})$. If $G_k(\lambda_1, \alpha_1) = 0$, then one obtains a k-periodic solution x^* of (3.1) and it can be shown that the sign of $\partial G(\lambda_1, \alpha_1)/\partial\alpha$ is the same as the sign of the characteristic exponent near zero of the linear variational equation for x^*. If (λ_1, α_1), (λ_2, α_2) are the roots of $G_k(\lambda, \alpha)$ near $(\lambda_0, \alpha_M^{k0})$, $\alpha_1 < \alpha_2$, and x_1^*, x_2^* are the corresponding solutions, then

(i) x_1^* is a saddle if $\lambda_1 > 0$, x_2^* is a stable node;

(ii) x_1^* is an unstable node if $\lambda_1 < 0$, x_2^* is a saddle.

One can do the same argument near α_m^{k0} to obtain two solutions x_3^*, x_4^* corresponding to $\alpha_3 < \alpha_4$ such that

(iii) x_3^* os a stable node if $\lambda_1 > 0$, x_4^* is a saddle;

(iv) x_3^* is a saddle if $\lambda_1 < 0$, x_4^* is an unstable node.

This proves the lemma for values of λ near the bifurcation curves C_M, C_m for homoclinic orbits. To show the same results for any $\lambda \in S_2$, we proceed as follows.

If $G_k(\lambda_1, \alpha_1) = 0$ and $\partial G_k(\lambda_1, \alpha_1)/\partial\alpha \neq 0$, then the corresponding k-periodic solution x_0 of (3.1) is hyperbolic. If this were not so, we could make a small perturbation in (3.1) and obtain more than one k-periodic solution corresponding to (λ, α) near (λ_1, α_1). But this contradicts $\partial G_k(\lambda_1, \alpha_1)/\partial\alpha \neq 0$. Our hypothesis on h implies any change in the number of periodic solutions must occur in the same way as on the bifurcation curves C_m, C_M; that is, through the creation or annihilation of a saddle and a node. This clearly implies the assertion stated in the lemma.

This completes the proof of the lemma.

LEMMA 4.3. *Suppose the conditions of Lemma 4.2 are satisfied, and let*

$$\nu_M(\delta) = h(\alpha_M{}^\delta, \delta), \qquad \nu_m(\delta) = h(\alpha_m{}^\delta, \delta); \tag{4.9}$$

suppose $\nu_M'(0) \neq 0$, $\nu_m'(0) \neq 0$, and let the sectors S_1, S_2 be defined as in Lemma 3.1. Then the following conclusions hold:

(i) *If $\nu_M'(0) > 0$, then there exist infinitely many subharmonic bifurcations as one goes from S_1 to S_2 through the bifurcation curve C_M and none going from S_2 to S_1 through C_M. If $\nu_M'(0) < 0$, the opposite situation prevails.*

(ii) *If $\nu_m'(0) < 0$, then there exist infinitely many subharmonic bifurcations as one goes from S_1 to S_2 through the bifurcation curve C_m and none going from S_2 to S_1 through C_m. If $\nu_m'(0) > 0$, the opposite situation prevails.*

Remark. The derivatives above are explicitly obtained from the following formula:

$$\frac{\partial h(\alpha, 0)}{\partial \delta} = \int_{-\infty}^{\infty} \frac{\partial p_\delta(t)}{\partial \delta}\bigg|_{\delta=0} f(t-\alpha)\, dt,$$

$$\begin{aligned} \frac{\partial p_\delta(t)}{\partial \delta}\bigg|_{\delta=0} &= -\frac{g'(p_0(0))\, p_0(0)}{g(p_0(0))}\, \dot{p}(t) \\ &\quad - \int_0^t [\dot{p}(s)\, q(t) - \dot{p}(t)\, q(s)]\, g''(p_0(s))\, \dot{p}_0(s)\, ds. \end{aligned}$$

Proof. This is a consequence of Lemma 3.3 and the fact that all bifurcation curves $C_m{}^k$, $C_M{}^k$ are determined by the minimum and maximum of $h(\alpha, \delta_k)$.

All of the previous results are summarized in the following theorem.

THEOREM 4.1. *If $h(\alpha)$ in (3.2) satisfies (3.3), and there are only a finite number of points $\alpha_j \in [0, 1)$ where $h'(\alpha_j) = 0$, and (4.4) is satisfied, then there is a neighborhood U of Γ, V of $\lambda = 0$ and two C^2-curves $C_M, C_m \subseteq V$ such that $V - (C_M \cup C_m) = S_1 \cup S_2$ such that the following conclusions hold:*

(i) *there exist no homoclinic points of (3.1) for any $\lambda \in S_1$;*

(ii) *there exists a transverse homoclinic point of (3.1) for any $\lambda \in S_2$;*

(iii) *there are infinitely many subharmonic bifurcations in any neighborhood of any point on the bifurcation curves C_M, C_m to homoclinic points;*

(iv) *the subharmonic bifurcations occur before or after homoclinic bifurcation according to Lemma 3.4;*

(v) *for any $\lambda \in S_2$ there are infinitely many hyperbolic saddles and nodes or foci of (3.1) corresponding to fixed points of iterates of the time one map.*

5. An Example

In this section, we give an example for which the conditions (1.3) are satisfied. The authors are indebted to K. Meyer and D. Henry for the computations. Consider the equation

$$\ddot{x} - x + \tfrac{3}{2}x^2 = 0 \tag{5.1}$$

which admits the first integral

$$I = \dot{x}^2 - x^2 + x^3. \tag{5.2}$$

Let $p(t)$ be the solution of (5.1) such that $p(0) = 1$, $\dot{p}(0) = 0$ so $I(p) = 0$. The function p is even, $\dot{p}$ is odd, $p > 0$ and p is decreasing on $(0, \infty)$.

The relation $I(p) = 0$ implies $dp/dt = -p(1-p)^{1/2}$ for $t > 0$. An elementary integration implies

$$p(t) = 1 - \left(\frac{1-e^t}{1+e^t}\right)^2.$$

Let $f(t) = \cos t$ and define the function $h(\alpha)$ in (1.2),

$$\begin{aligned} h(\alpha) &= \int_{-\infty}^{\infty} \dot{p}(t)\cos(t-\alpha)\,dt \\ &= -\int_{-\infty}^{\infty} p(t)\sin(t-\alpha)\,dt \\ &= -\sin\alpha \int_{-\infty}^{\infty} p(t)\cos t\,dt - \cos\alpha \int_{-\infty}^{\infty} p(t)\sin t\,dt \\ &= -\Delta\sin\alpha, \end{aligned}$$

where $\Delta = \int_{-\infty}^{\infty} p(t)\cos t\,dt$. Since $p(t) = \operatorname{sech}^2(t/2)$, we have

$$\begin{aligned} \Delta &= 2\textstyle\int_{-\infty}^{\infty} e^{2i\tau}/(\cosh^2\tau)\,d\tau = 4\pi i \text{ (sum of residues in the upper half plane)} \\ &= 8\pi\omega \textstyle\sum_{n\geqslant 0} e^{-2\pi(n+1/2)} = 4\pi\omega/\sinh\pi. \end{aligned}$$

Thus, $\Delta > 0$.

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